3rd ESO
Chapter 1: Rational Numbers

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BOCM: – Some irrational numbers in everyday life situations. The importance of the number π and the golden ratio. – Order on the number line. Representation of irrational numbers on it. Intervals (open, closed, mixed, and half-lines).

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Contents

- 0. WHAT YOU SHOULD REMEMBER
  - 0.1. Priority of Operations
  - 0.2. Use of Parentheses
  - 0.3. Operations with Integers
- 1. RATIONAL NUMBERS
  - 1.1. Definition
  - 1.2. Equivalent Fractions
  - 1.3. Ordering Fractions
  - 1.4. Representation on the Number Line
  - 1.5. Operations with Fractions
  - 1.6. Irrational Numbers. Decimal Expression of Irrational Numbers
  - 1.7. Different Types of Numbers
- 2. APPROXIMATIONS AND ERRORS
  - 2.1. Rounding
  - 2.2. Significant Figures
  - 2.3. Absolute Error and Relative Error
- 3. FRACTIONS AND DECIMALS
  - 3.1. Decimal Expression of a Fraction
  - 3.2. Fraction Form of a Decimal Expression. Generating Fraction
- 4. INTERVALS, HALF-LINES, AND NEIGHBORHOODS
  - 4.1. Intervals. Types and Meaning
  - 4.2. Half-lines
  - 4.3. Neighborhoods
- 5. PROBLEM SOLVING USING FRACTIONS

Summary
In this chapter we will recall many things you already know from previous years, such as operations with natural numbers and integers, operations with fractions and decimal expressions. We will study rational numbers, and begin to learn something about irrational numbers.

0. WHAT YOU SHOULD REMEMBER

0.1. Priority of Operations

When there are no parentheses to indicate which operation to perform first, or inside parentheses, an agreement has been reached on how to proceed. Namely:

1. Solve inner parentheses. If there are no parentheses or inside parentheses, we will do:
2. Powers and roots.
3. Multiplications and divisions.
4. Additions and subtractions.

Expressions such as $1 - 100 : 5 \cdot 5$, where it is not clear what to do (multiplication and division have the same priority), should be avoided. Parentheses should be used to indicate which to do first. The above expression can be:

$$1 - (100 : 5) \cdot 5 = -99 \qquad \text{or} \qquad 1 - 100 : (5 \cdot 5) = -3.$$

Nevertheless, if you encounter it, you will do:

5. If there are several operations with the same priority, they will be done from left to right.

Examples:

- $(5 - 7) \cdot 10 - 8$ We cannot do $10 - 8$ (well you can, but you shouldn't)

  First the parentheses → $-2 \cdot 10 - 8$ Then the product → $-20 - 8$ Finally the subtraction → -28
- $10 - 2 \cdot 3^2 = 10 - 2 \cdot 9 = 10 - 18 = -8.$ Here it is forbidden to do $10 - 2$ and $2 \cdot 3$.
- $3 \cdot (-2 + 4)^2 - 8 - 5 \cdot 2^2 = 3 \cdot 2^2 - 8 - 5 \cdot 4 = 12 - 8 - 20 = -16$
- -10^2 equals -100 since the exponent is applied first, and the minus sign is not squared. However, $(-10)^2$ does equal $+100$.

$$-10^2 = -10 \cdot 10 = -100$$

$$(-10)^2 = (-10) \cdot (-10) = +100$$

- $\sqrt{9} \cdot 25 = 3 \cdot 25 = 75$. The root is performed first.
- $10 - 9x$ is not $1x$ since subtraction cannot be performed under any circumstances.

Note that this priority is always valid, for operations with all types of numbers or other objects (e.g., polynomials). It's worth learning, isn't it?

0.2. Use of Parentheses

Parentheses indicate the operations that must be done first. In fact, the first thing we do is the innermost parentheses, proceeding from inside out. It's like getting dressed: first you put on a T-shirt, then a sweater, then a jacket. Doing it the other way round is complicated. Therefore, before you start calculating wildly, look at the whole expression to see what to do first.

- There must be as many open parentheses as closed; otherwise, they are said to be "unbalanced parentheses".
- If something multiplies a parenthesis, it is not necessary to put the "·" symbol.

Examples:

$$2 \cdot (2 - 2 \cdot (2 - 2 \cdot 2)) = 2 \cdot (2 - 2 \cdot (2 - 4)) = 2 \cdot (2 - 2 \cdot (-2)) = 2 \cdot (2 + 4) = 2 \cdot 6 = 12$$

$$2(3 - 2) = 2 \cdot 1$$

$$(2 - 3) \cdot (6 - 4) = -1 \cdot 2 = -2$$

If we want to divide by 2 the result of $75 - 90$, we should not write $75 - 90 : 2$; here the 2 only divides 90. We write $(75 - 90) : 2$.

Parentheses are used to enclose function arguments. For example:
If in a program or calculator we want to take the square root of $100 \cdot 3^4$, we write `sqrt(100*3^+4)`.

0.3. Operations with Integers

We recall the most important points:

Sign rules for addition:

- The sum of two positive numbers is positive. Example: $+5+7=+12$
- The sum of two negative numbers is negative. Example: $-10-17=-27$
  Write the sign $-$, and add their absolute values.
  Example: If I lose 10 and then lose another 17, I have lost 27.
- The sum of a positive and a negative number will have the sign of the larger absolute value.
  Example: $-7+15=+8$; $+8+(-20)=8-20=-12$
  Write the sign of the larger (in absolute value) and subtract.
  Example: If I lose 7 and then win 15, I win 8 (gains exceed losses).
  Example: If I win 8 but then lose 20, I lose 12 (losses exceed gains).

Sign rules for multiplication (and division):

- Positive $\times$ Positive = Positive
- Positive $\times$ Negative = Negative $\times$ Positive = Negative
- Negative $\times$ Negative = Positive.

Examples:
$+2\cdot(-7)=-14$. If I inherit 2 debts of €7, I have a debt of €14.
$-2\cdot(-7)=+14$. If they remove 2 debts of €7 from me, I have gained €14!

Now some serious mathematics, since we are in 3rd year!

Rigorous proof that "$0\cdot x=0$ for all x" and that "$(-1)\cdot(-1)=+1$"

To do this we will use 4 properties of numbers that you know:

1. $a+0=a$ for every number a (0 is the identity element for addition)
2. The distributive property: $a\cdot(b+c)=a\cdot b+a\cdot c$
3. $1\cdot a=a$ for every number a (1 is the identity element for multiplication)
4. $-a$ is the opposite of $+a$, that is, $-a+a=a+(-a)=0$

We prove "$0\cdot x=0$ for every number x":

$$\text{Since } a-a=0, \quad x(a-a)=x\cdot 0=xa-xa=0$$

We prove that "$(-1)\cdot(-1) = +1$":

$$(-1)\cdot(-1+1) = (-1)\cdot 0 = 0; \quad \text{but by the distributive property}$$
$$(-1)\cdot(-1+1) = (-1)\cdot(-1) + (-1)\cdot 1 = (-1)\cdot(-1) + (-1).$$

Thus $(-1)\cdot(-1) + (-1) = 0$.

If we add 1 to both sides:

$$(-1)\cdot(-1) + (-1) + 1 = +1 \ \rightarrow\ (-1)\cdot(-1) + 0 = +1 \ \rightarrow\ (-1)\cdot(-1) = +1$$

Solved activities

Calculate step by step:

$$(((-15 - 5\cdot(-20-6)) : (15 - 4^2)) + 5 - 4\cdot 2)\cdot(-10)$$

First compute $-20 - 6 = -26$; $4^2 = 16$ and $4\cdot 2 = 8$, and we get:

$$\begin{aligned}
& (((-15 - 5\cdot(-26)) : (15 - 16)) + 5 - 8)\cdot(-10) \\
&= (((-15 + 130) : (-1)) - 3)\cdot(-10) \\
&= ((115 : (-1)) - 3)\cdot(-10) \\
&= (-115 - 3)\cdot(-10) \\
&= -118\cdot(-10) = +1\,180
\end{aligned}$$

Proposed activities

1. Calculate:
   (a) $-20 + 15$
   (b) $-2\cdot(-20+15)$
   (c) $-20 : (10 - 2(-20+15))$
   (d) $(-80 - 20 : (10 - 2(-20+15)))\cdot(3 - 2\cdot 3^2)$

2. Calculate:
   (a) $-10 + 20 : (-5)$
   (b) $(-10 + 20) : (-5)$
   (c) $-100 : ((-20) : (-5))$
   (d) $(-100 : (-20)) : (-5)$
   (e) $\sqrt{36}\cdot 4$

3. Calculate:

   (a) $3 - (4 \cdot 3 - 2 \cdot 5)^2 - (3 - 5)^3$
   (b) $5 - 3^2 - 2 \cdot (-5) - (7 - 9)^2$
   (c) $7 - 2 \cdot (3 - 5)^2 + 2 \cdot (-3) + 8 - (-2)^2$
   (d) $2 - (2 \cdot 3 - 3 \cdot 4)^2 - (2 - 4)^3$

1 RATIONAL NUMBERS

1.1 Definition

Rational numbers are all those numbers that can be expressed as a fraction of integers. That is, a number r is rational if $r = \frac{a}{b}$, with a, b integers and $b \neq 0$.

A fraction is an indicated division, for example $\frac{7}{3}$ means $7 : 3$, but the division is not carried out until we need it. There are many occasions when it is better to leave the operations indicated.

One example is enough:
Try to do the division $1.142857142857\ldots : 8$. Difficult, right? However, $\frac{8}{7} : 8 = \frac{1}{7}$ is somewhat easier and also exact.

The name "rational" comes from "ratio", which in mathematics means division or quotient.

The set of rational numbers is represented by $\mathbb{Q}$.

A rational number has infinitely many representations as a fraction.
Thus: $\frac{1}{3} = \frac{3}{9} = \frac{6}{18} = \ldots$ are infinitely many fractions that represent the same rational number; they are called "equivalent" because they have the same numerical value. If we perform the divisions in the example, all of them equal $0.333\ldots$, which is their decimal expression.

"Integers" are rational because they can be expressed as a fraction, for example $-2 = \frac{-8}{4}$.

Every rational number has a representative that is its irreducible fraction, the one with the smallest possible numbers in the numerator and denominator. This fraction is reached from any other by dividing the numerator and denominator by the same number. If you want to do it in a single step, divide by the Greatest Common Divisor (GCD) of the numerator and denominator. For example: $\frac{60}{80} = \frac{6}{8} = \frac{3}{4}$ where we first divided by 10 and then by 2, but we could have divided directly by 20 since 20 is the GCD of 60 and 80. Therefore $\frac{3}{4}$ is the irreducible fraction and hence the one that represents the rational number that has many other fractional forms such as $60/80 = 6/8 = 30/40 = 12/16 = 9/12 = 15/20 = 18/24 = 21/28 = 24/32 = 27/36\ldots$ and a decimal expression of 0.75.

1.2 Equivalent Fractions

Two fractions are equivalent if the following conditions hold (all of them equivalent):

- **By performing the division we obtain the same decimal expression.** This is the definition.
  Example: $4:5 = 8:10 = 0.8$, so $\frac{4}{5}$ and $\frac{8}{10}$ are equivalent and we may write $\frac{4}{5} = \frac{8}{10}$.

- **Cross products are equal:** $\frac{a}{b} = \frac{c}{d} \iff a \cdot d = b \cdot c$.
  It is easy to prove: multiply both sides of the equality by b and by d:
  $$\frac{a}{b} \cdot b \cdot d = \frac{c}{d} \cdot b \cdot d \quad \Longrightarrow \quad a \cdot d = c \cdot b.$$
  Example: $\frac{12}{8} = \frac{6}{4}$ since $12 \cdot 4 = 8 \cdot 6 = 48$.

- **When simplified, the fractions lead to the same irreducible fraction.** If $\frac{A}{B} = \frac{C}{D}$ and both reduce to the same irreducible fraction, then they are equivalent.
  Example: $\frac{80}{60} = \frac{4}{3}$ and $\frac{12}{9} = \frac{4}{3}$, thus $\frac{80}{60} = \frac{12}{9}$.

- **One fraction can be obtained from the other by multiplying (or dividing) the numerator and denominator by the same number.**
  Example: $\frac{6}{4} = \frac{24}{16}$ because it suffices to multiply the numerator and denominator of the first by 4 to obtain the second.
  In general: $\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$.

Reduction to a Common Denominator

To compare two or more fractions (to see which is larger) and also to add or subtract them, it is important to obtain equivalent fractions that have the same denominator.

First an example and then the theory:

We want to compare $\frac{5}{6}$ and $\frac{6}{7}$. The least common multiple of 6 and 7 is 42. We put 42 as the new denominator for both fractions and calculate the new numerators so that the fractions are equivalent:

$$\frac{5}{6} = \frac{5 \cdot 7}{6 \cdot 7} = \frac{35}{42}, \qquad \frac{6}{7} = \frac{6 \cdot 6}{7 \cdot 6} = \frac{36}{42}.$$

Now it is clear which of the two is larger: $\frac{36}{42} > \frac{35}{42}$, so $\frac{6}{7} > \frac{5}{6}$.

1.3 Ordering Fractions

To sort a set of fractions there are several methods:

i) Perform the divisions and compare the decimal expressions.

This method is the easiest but not the fastest (unless you have a calculator).

Example: We are asked to order the following fractions from smallest to largest:

$$\frac{20}{19}, \quad \frac{21}{20}, \quad -\frac{20}{19}, \quad -\frac{21}{20}, \quad \frac{29}{30}, \quad \frac{28}{29}.$$

Performing the divisions we obtain respectively: $1.0526\ldots$; 1.05; $-1.0526\ldots$; -1.05; $0.9666\ldots$; and $0.9655\ldots$. Looking at the decimal numbers we see:

$$-\frac{20}{19} < -\frac{21}{20} < \frac{28}{29} < \frac{29}{30} < \frac{21}{20} < \frac{20}{19}.$$

Remember: Negative numbers are always smaller than positive ones, and among negative numbers the one with the larger absolute value is smaller ($-4 < -3$).

ii) Use logic and the following trick: for positive fractions $\frac{a}{b} < \frac{c}{d} \iff a \cdot d < b \cdot c$.

Example: $\frac{8}{10} < \frac{9}{11}$ since $8 \cdot 11 < 9 \cdot 10$.

$$8 \cdot 11 < 9 \cdot 10 \implies \frac{8 \cdot 11}{9 \cdot 10} < 1 \implies \frac{8}{10} \cdot \frac{11}{9} < 1 \implies \frac{8}{10} < \frac{9}{11},$$

where we multiplied by $9 \cdot 11$ and simplified. Conversely, if $\frac{8}{10} < \frac{9}{11}$ then multiplying by $10 \cdot 11$ yields $8 \cdot 11 < 9 \cdot 10$. And what is meant by "using logic"? We start with the easiest.

Example: Compare $\frac{20}{19}$ and $\frac{28}{29}$.

$\frac{20}{19} > 1$ because $20 > 19$. But $\frac{28}{29} < 1$ because $28 < 29$. Clearly the second is smaller.

A little harder: compare $\frac{20}{19}$ and $\frac{21}{20}$.

$$\frac{20}{19} = \frac{19+1}{19} = 1 + \frac{1}{19}, \qquad \frac{21}{20} = \frac{20+1}{20} = 1 + \frac{1}{20}.$$

Which is larger, $\frac{1}{19}$ or $\frac{1}{20}$? $\frac{1}{19}$ is larger, so the first fraction is larger. Think: if you divide a pizza into 19 equal slices, they are larger than if you divide it into 20 equal slices.
If a and b are positive, $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$. So $\frac{1}{3} > \frac{1}{4}$, for example.

Even harder: compare $\dfrac{19}{20}$ and $\dfrac{18}{19}$. Now $\frac{19}{20} = 1 - \frac{1}{20}$ and $\frac{18}{19} = 1 - \frac{1}{19}$.
Since $\frac{1}{19} > \frac{1}{20}$, the fraction $\frac{19}{20}$ is larger because it is missing less to reach 1.
With simpler numbers: $\frac{2}{3} < \frac{3}{4}$ because $\frac{2}{3}$ lacks $\frac{1}{3}$ to reach 1, while $\frac{3}{4}$ lacks only $\frac{1}{4}$.
Important: If a and b are positive, $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$.

iii) Reduce to a common denominator and compare the numerators:

We are asked to order the following fractions from largest to smallest:

$$\frac{5}{6}, \quad \frac{7}{8}, \quad -\frac{9}{4}, \quad -\frac{7}{3}, \quad -2.$$

First we find a number that is a multiple of $6, 8, 4$, and 3 (the least common multiple is best). We find 24, which is a multiple of all of them. We put it as the new denominator for all fractions and calculate the new numerators so that the fractions remain equivalent:

$$\frac{5}{6} = \frac{5\cdot 4}{6\cdot 4} = \frac{20}{24}, \quad \frac{7}{8} = \frac{7\cdot 3}{8\cdot 3} = \frac{21}{24}, \quad -\frac{9}{4} = \frac{-9\cdot 6}{4\cdot 6} = \frac{-54}{24}, \quad -\frac{7}{3} = \frac{-7\cdot 8}{3\cdot 8} = \frac{-56}{24}, \quad -2 = \frac{-2\cdot 24}{1\cdot 24} =$$

Then we compare the numerators and obtain:

$$\frac{21}{24} > \frac{20}{24} > \frac{-48}{24} > \frac{-54}{24} > \frac{-56}{24} \quad \Longrightarrow \quad \frac{7}{8} > \frac{5}{6} > -2 > -\frac{9}{4} > -\frac{7}{3}.$$

1.4 Representation on the Number Line

This is the number line; on it every real number has an exact position.

We recall things you already know:

- To draw it you can only make two decisions: where to place 0 and where to place 1; i.e., where the origin is and what the size of the unit is.
- The units must always be of the same size.
- Positive numbers go to the right of 0 and negative numbers to the left.
- 0 is neither positive nor negative.
- The number line has no beginning and no end. We can only draw a “small” part of it.

- Given two numbers a, b, we have $a < b$ if a is to the left of b, and vice versa. For example: $1 < 3$; $-1 < 1$; $-4 < -2$.

Every rational number has a predetermined position on the number line. The infinitely many equivalent fractions that make up a rational number fall on the same point of the line. So $\frac{2}{3}$ and $\frac{4}{6}$, which are the same number, fall on the same point.

We will see how to represent fractions exactly.

Proper Fraction, Improper Fraction, and Mixed Form

Proper fraction: A fraction $\frac{a}{b}$ where $a < b$. That is, the numerator is smaller than the denominator.
For example: $\frac{4}{5}$ or $\frac{99}{100}$.
If $a < b$, upon division the decimal expression will be less than 1.
For example: $\frac{4}{5} = 4 : 5 = 0.8$.

Improper fraction: A fraction $\frac{a}{b}$ where $a > b$, numerator greater than the denominator.
Example: $\frac{15}{4}$ or $\frac{37}{27}$. If we divide, the decimal expression is greater than 1. $\frac{15}{4} = 3.75$ and $\frac{37}{27} = 1.37037037\ldots$

Mixed number: Improper fractions can be written as the sum of an integer and a proper fraction.
Thus, for example: $\frac{9}{5} = \frac{5+4}{5} = 1 + \frac{4}{5}$; this last form is the mixed form.
In Spain this is not common, but in the English-speaking world it is usually written as $1\frac{4}{5}$, which means the same thing.
Scientific calculators convert to mixed form—look into it.

The quick and automatic way to write a fraction in mixed form is as follows: $\frac{77}{6}$ is improper since $77 > 6$; to write it in mixed form we perform the integer division $77 : 6$, that is, without decimals—we are interested in the quotient and the remainder.

The quotient is the integer part, the remainder is the numerator of the fraction, and the divisor is the denominator.

It is important that you try to do it mentally (when reasonable). It is easy, for example:
$\frac{47}{6}$: we look for the multiple of 6 closest to 47 from below, which is $7 \cdot 6 = 42$, therefore:

$$\frac{47}{6} = 7 + \frac{5}{6}$$

since from 42 to 47 there are 5. Think about it: if we eat $\frac{47}{6}$ of a pizza, we have eaten 7 whole pizzas and $\frac{5}{6}$ of a pizza.

Note:

It is also easy to find the quotient and remainder with a calculator, in case you are in a hurry.
For $\frac{437}{6}$, do the division $437 : 6$; you get $72.83333\ldots$; the integer part is 72, we only need to calculate the remainder. We have two paths:

1. Do $437 - 72 \cdot 6 = 5$, and that's it.
2. Multiply the decimal part by the divisor: $0.8333\ldots \cdot 6 = 5$, which is the remainder. If necessary, round ($0.8333 \cdot 6 = 4.9998$, which we round to 5). We only allow you to do this if you know why it works; if not, forget it.

If the fraction is negative, we proceed as follows:

$$-\frac{19}{5} = -3 - \frac{4}{5}$$

since the division gives a quotient of 3 and a remainder of 4.

Representation of Fractions

a) If the fraction is proper:

Example: Represent the fraction $\frac{5}{6}$. Its value is between 0 and 1, therefore we divide the first unit into 6 equal parts and take 5.

In the figure it is shown how to do it exactly using Thales' Theorem. We draw an oblique line passing through 0, mark 6 points with the compass at equal distances (whatever distance, but equal). We join the last point with 1 and draw lines parallel to that segment passing through the intermediate points of the oblique line (dashed lines). These parallel lines divide the interval [0, 1] into 6 equal parts.

Note that to divide into 6 equal parts you only need to mark 5 intermediate points at equal distances—always one less. To divide into 8 equal parts, we mark 7 intermediate points.

If the fraction is negative, do the same but in the interval $[-1, 0]$.
In the figure we have represented $-\frac{5}{8}$; we divided the interval $[-1, 0]$ into 8 equal parts and counted 5 starting at 0. Make sure you understand it, and if not, ask. By the way, the arrow points to the point and not to the space between them.

If we want to represent the proper fraction $\frac{a}{b}$, divide the first unit into "b" equal parts and count "a" divisions.
In the case of a negative fraction, do the same but counting from 0 to the left.

b) If the fraction is improper:

Solved activities

We represent $\frac{13}{6}$. First write it in mixed form: $2\frac{1}{6}$, now it is easy to represent: we go to 2, divide the unit from 2 to 3 into 6 equal parts and take 1 (see image).

Similarly for $\frac{11}{8} = 1 + \frac{3}{8}$: we go to 1 and divide the unit from 1 to 2 into 8 equal parts and take 3.

If the fraction is negative, we proceed as follows:
We represent $-\frac{12}{7} = -1 - \frac{5}{7}$: we go to -1, divide the unit from -1 to -2 into 7 equal parts and count 5 to the left starting from -1.
We represent $-\frac{11}{4} = -2 - \frac{3}{4}$: we go to -2, divide into 4 equal parts and take 3, counting to the left and starting from -2 (see image).

Proposed Activities

1. Write the following fractions in mixed form: $\dfrac{50}{7}$; $\dfrac{25}{11}$; $\dfrac{101}{6}$
2. Write the following fractions in mixed form: $-\dfrac{30}{7}$; $-\dfrac{50}{13}$; $-\dfrac{100}{21}$
3. Represent on the number line the fractions: $\dfrac{1}{5}$; $-\dfrac{3}{7}$; $\dfrac{5}{8}$; $-\dfrac{3}{4}$
4. Write in mixed form and represent the fractions: $\dfrac{23}{8}$; $-\dfrac{23}{8}$; $\dfrac{180}{50}$; $-\dfrac{26}{6}$
5. Find the fractions corresponding to points A, B, C, D, and E, expressing those represented by points A, B, and E in mixed form and as improper fractions.

1.5 Operations with Fractions

We will review operations with fractions, specifically addition, subtraction, multiplication, and division.

Addition and Subtraction of Fractions

Addition and subtraction are the most demanding operations because you can only add or subtract like things. We cannot add meters and seconds, nor euros and liters. In the same way, thirds cannot be added to fifths, nor fourths to halves. That is, we cannot directly perform the sum $\frac{5}{6} + \frac{3}{4}$ because sixths and fourths are of different sizes. But is there a way to add them? Yes.

First, find two equivalent fractions that have the same denominator, and then they can be added. Let's look at an example:

A multiple of 6 and 4 is 12. We write 12 as the new denominator and find the numerators so that the fractions are equivalent: $\frac{5}{6} = \frac{5\cdot 2}{6\cdot 2} = \frac{10}{12}$ and $\frac{3}{4} = \frac{3\cdot 3}{4\cdot 3} = \frac{9}{12}$. Now we can add the twelfths, and the result is in twelfths:

$$\frac{10}{12} + \frac{9}{12} = \frac{10+9}{12} = \frac{19}{12}.$$

Another example: $\dfrac{13}{6} - \dfrac{51}{10} + \dfrac{8}{12}$

$$\frac{13\cdot 10}{6\cdot 10} - \frac{51\cdot 6}{10\cdot 6} + \frac{8\cdot 5}{12\cdot 5} = \frac{130}{60} - \frac{306}{60} + \frac{40}{60} = \frac{130-306+40}{60} = \frac{-136}{60} = -\frac{34}{15}.$$

We found a multiple of 6, 10, and 12 (if it is the least common multiple, so much the better), wrote it as the common denominator and did $60 : 6 = 10$, so we multiplied the 13 by 10; $60 : 10 = 6$, so we multiplied the 51 by 6, etc. When all fractions have the same denominator, add or subtract the numerators, keeping the same denominator. If possible, simplify the resulting fraction.

In cases where it is not easy to find the least common multiple, do:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a\cdot d \pm c\cdot b}{b\cdot d}.$$

For example:

$$\frac{15}{387} + \frac{19}{155} = \frac{15\cdot 155 + 19\cdot 387}{387\cdot 155} = \frac{9\,678}{59\,985} = \frac{3\,226}{19\,995}.$$

Multiplication and Division of Fractions

It is surprising that multiplication and division of fractions are simpler than addition and subtraction.

Multiplication: $\dfrac{a}{b}\cdot\dfrac{c}{d} = \dfrac{a\cdot c}{b\cdot d}$; multiply the numerators together to obtain the numerator of the product fraction, and the denominators together to determine the denominator of that fraction—easy, right?

Thus:

$$\frac{3}{11}\cdot\frac{5}{7} = \frac{3\cdot 5}{11\cdot 7} = \frac{15}{77}.$$

Why are fractions multiplied this way? We will not prove the general case; one example will suffice.

$$\frac{2}{3}\cdot\frac{4}{5}$$

means $\frac{2}{3}$ *of* $\frac{4}{5}$. The product is $\frac{8}{15}$ out of the 20 total.

Sometimes it is convenient to multiply intelligently: $\frac{17}{3} \cdot \frac{15}{17} = \frac{\not{17}}{3} \cdot \frac{15}{\not{17}} = \frac{15}{3} =$ 5Before multiplying, notice that the 17 can be canceled (why multiply by 17 and then divide by 17?) and then the 5, since $15 = 3 \cdot 5$.

Another example:

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6}$$

Do it; we hope you arrive at the correct simplified result, which is $\frac{1}{6}$ ☺.

We have something important to tell you: we never, ever want to see this:

$$\frac{x}{x+5} = \frac{\not{x}}{\not{x}+5} = \frac{1}{1+5} = \frac{1}{6}$$

is absolutely false ($\frac{10}{12} = \frac{5}{6}$ is correct). You can only cancel if the number is multiplying in both the numerator and the denominator (if it is a common factor). This is also very wrong:

$$\frac{x \cdot 2 + 3}{x \cdot 4 + 5} = \frac{2+3}{4+5}.$$

Fraction inverse

The inverse fraction of $\dfrac{a}{b}$ is $\dfrac{b}{a}$ because it satisfies $\dfrac{a}{b} \cdot \dfrac{b}{a} = \dfrac{ab}{ba} = 1$, which is the definition of inverse.

Examples: The inverse of $\frac{3}{4}$ is $\frac{4}{3}$ and the inverse of 2 is $\frac{1}{2}$.

Division

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc}$$

Thus, to divide, multiply by the inverse of the divisor fraction.

For example:

$$\frac{6}{10} : \frac{12}{15} = \frac{6}{10} \cdot \frac{15}{12} = \frac{2 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 5 \cdot 2 \cdot 2 \cdot 3} = \frac{3}{4}$$

You can also multiply and then simplify: $\dfrac{90}{120} = \dfrac{3}{4}$.

You may ask if you can cross-multiply—it depends on your teacher.

Curious cases:

- Dividing by one tenth is multiplying by 10, since $a : \dfrac{1}{10} = a \cdot 10 = 10a$.
- In general: dividing by $\frac{1}{a}$ is multiplying by a.

- Dividing by a number is like multiplying by its inverse: $a : 2 = a \cdot \dfrac{1}{2} = \dfrac{a}{2}$.
- Towers of fractions: Do not be scared if you see something like $\frac{\frac{10}{4}}{\frac{15}{6}}$, it is very easy: it means $\frac{10}{4} : \frac{15}{6} = \frac{10}{4} \cdot \frac{6}{15} = \frac{60}{60} = 1$. Don't forget that the fraction bar means the same as ":".

Combined operations

We apply everything we "know" about priority and use of parentheses.

Solved activities

Calculate step by step and simplify:

$$\left(\frac{3}{4} - \left(\frac{1}{2} - \frac{4}{6}\right)\right) \cdot \left(\frac{1}{2} - \frac{3}{14} - \frac{2}{3}\right)$$

First, do the innermost parenthesis and the multiplication in the second parenthesis (which has priority over subtraction).

$$\left(\frac{3}{4} - \left(\frac{3}{6} - \frac{4}{6}\right)\right) \cdot \left(\frac{1}{2} - \frac{1}{7}\right) = \left(\frac{3}{4} - \left(-\frac{1}{6}\right)\right) \cdot \left(\frac{7}{14} - \frac{2}{14}\right)$$
$$= \left(\frac{3}{4} + \frac{1}{6}\right) \cdot \frac{5}{14} = \left(\frac{9}{12} + \frac{2}{12}\right) \cdot \frac{5}{14} = \frac{11}{12} \cdot \frac{14}{5} = \frac{154}{60} = \frac{77}{30}$$

Fraction as an operator

a) Fraction of a number:

We are asked to find three quarters of 120. We translate: find $\frac{3}{4}$ of 120. This "of" becomes a "times" in mathematics, so:

$$\frac{3}{4} \text{ of } 120 = \frac{3}{4} \cdot 120 = \frac{3 \cdot 120}{4} = 3 \cdot 30 = 90.$$

In general: $\dfrac{a}{b}$ of $c = \dfrac{a}{b} \cdot c = \dfrac{ac}{b}$.

b) Fraction of a fraction:

Examples:

$$\frac{10}{6} \text{ of } \frac{4}{15} = \frac{10}{6} \cdot \frac{4}{15} = \frac{40}{90} = \frac{4}{9}.$$
$$\frac{2}{5} \text{ of } \frac{10}{12} \text{ of } 360 = \frac{2}{5} \cdot \frac{10}{12} \cdot 360 = \frac{20 \cdot 360}{60} = 20 \cdot 6 = 120.$$

c) Inverse problem:

If $\frac{3}{4}$ of a number is 66, what is the number?

It is clear that one quarter is $66 : 3 = 22$ and the four quarters are $22 \cdot 4 = 88$.

In short: $66 \cdot \dfrac{4}{3} = 88$.

The general case: $\dfrac{a}{b}x = c \Rightarrow x = c \cdot \dfrac{b}{a}$; multiply the number by the inverse fraction.

Proposed activities

1. Find four fifths of three quarters of 12.
2. Five sixths of a number is 100. What is the number?

2 Irrational numbers. Decimal expression of irrational numbers

There are other numbers whose decimal expression is infinite and non-repeating. You already know some: π, $\sqrt{2}$, ... When the Greeks proved that numbers like $\sqrt{2}$, or the golden number, could not be expressed as a fraction and therefore had infinitely many non-repeating decimal digits, they found this extraordinary. That is why these numbers received the strange name "irrational". They could not understand it within their philosophy. The interesting thing is that there exists a length that measures exactly $\sqrt{2}$, namely the diagonal of a square of side 1, or the hypotenuse of an isosceles right triangle with legs 1.

The method to prove that $\sqrt{2}$ cannot be written as a fraction is called "reduction to the absurd". It consists in supposing that it can be, and reaching a contradiction. This procedure works for all non-exact roots, such as $\sqrt{3}, \sqrt{5}, \ldots$ But it does not work for all irrational numbers. Proving that π is irrational requires much more study. It is related to the interesting problem of squaring the circle. It was proved at the end of the 18th century by Lambert. Until then, people kept calculating decimals to find a period that does not exist.

Numbers whose decimal expression is infinite and non-repeating are called **irrational numbers**.

The set formed by the rational numbers and the irrational numbers is called the **real numbers**. With these numbers we have solved the problem of being able to measure any length. This property of the real numbers is called **completeness**. Each real number corresponds to a point on the line, and each point on the line corresponds to a real number. Note that each rational number also corresponds to a point on the line, but not the other way around, because $\sqrt{2}$ is a point on the line that is not rational.

3 Different types of numbers

You already know several types of numbers:

- **Natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$ – They are the numbers used for counting and ordering. 0 is not usually considered a natural number.

- **Integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ – They are the natural numbers, their opposites and zero. They have no decimal part, hence their name. They include the natural numbers.

- **Rational numbers** $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\right\}$ – Numbers that can be expressed as a quotient of two integers. Rational numbers include the integers. They also contain numbers that have a terminating decimal expression (e.g., 0.12345) and those with a repeating decimal expression (e.g., 7.01252525 . . .), because they can be written as fractions.

- **Irrational numbers** $\mathbb{I} = \mathbb{R} \backslash \mathbb{Q}$ – Numbers that cannot be expressed as a fraction of integers. They are those with an infinite, non-repeating decimal expression. Examples: 17.6766766676 . . . (just invented), or 0.1234567891011 . . . (invented by Carmichael). There are more irrational numbers than rational ones!

- **Real numbers** $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ – The union of rational and irrational numbers.

Thus we have:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}, \qquad \mathbb{I} \subset \mathbb{R}$$

Are these all the numbers? No, the reals are part of a larger set, the **Complex Numbers** $\mathbb{C}$ (studied in the first year of *Bachillerato* in the Science track).

4 Approximations and errors

In everyday life and in Applied Sciences it is necessary to work with approximate numbers. Some examples:

- We want to buy a third of a metre of fabric. We have to tell the shop assistant how much we want, and we are not going to be so foolish as to ask for 0.333 . . . metres or 33.333 . . . cm, which is the exact amount. Normally we ask for 33 cm or 333 mm if we are very precise.

- We measure an A4 sheet with a ruler and get 29.7 cm; the ruler is accurate to millimetres. We want to divide it into 8 equal parts. How much will each part measure? $29.7 : 8 = 3.7125$ cm, but the ruler cannot measure that precisely; it is better to approximate to 3.7 cm.

- We take an exam with 9 questions, all worth the same. We get 5 right and leave the rest blank. What mark do we get? $10 \cdot 5/9 = 5.555555556$ according to the calculator. Do we write all those digits? If we do, we are assuming we can distinguish one part in ten thousand million equal parts of the exam. It is reasonable to use 5.6 or 5.56 if we are very, very precise.

- It is curious, and should be a crime, that petrol stations announce: "Diesel price 1.399 €/litre". If someone asks for exactly one litre, or 2 or 15, they cannot charge that exact amount because thousandths of a euro do not exist! They should write 1.40 €/litre. It is true that you save 5 cents if you fill

> up 50 litres, but the psychological effect compensates them; people with little numerical culture see 1.3 instead of 1.4. Exactly the same happens in supermarkets: hake 5.99 €/kg. These are cheap tricks that a trained mind knows how to detect and act upon accordingly. The difference between 6 €/kg and 5.99 €/kg is that you save exactly 1 cent if you buy 1 kg; if you buy half a kilo, how much do you save? Nothing! $5.99 : 2 = 2.995$, which rounded is 3, exactly what they charge. Although, if you look at it properly, the offer is not so bad: if you buy 5 kg of hake you save enough to buy a sweet; of course, you have to buy more than half a kilo at a time.

Using too many decimal digits without being sure of them is not a sign of precision but of clumsiness.

4.1 Rounding

We remind you how to round numbers correctly.

Round π to the ten-thousandths: $\pi = 3.1415926535\ldots$, the ten-thousandths digit is 5; since the next digit is $9 \geq 5$, we add 1 to the 5 and write $\pi \approx 3.1416$. Note that π is closer to 3.1416 than to 3.1415.

Round $\sqrt{2}$ to the hundredths: $\sqrt{2} = 1.41421356\ldots$; the next digit is $4 < 5$, so we leave it as it is: $\sqrt{2} \approx 1.41$.

The rule is: Locate the rounding digit, look at the next digit (only the next one). If it is less than 5, leave the rounding digit unchanged; if it is 5 or greater than 5, increase the rounding digit by 1.

Example: 1.995 to hundredths: 2.00, and the zeros must be written to indicate where we rounded.

More examples:

- 1 555 555 to the thousands: 1 556 000, where we must fill with zeros after the thousands.
- 6.94999 to tenths: 6.9; you only need to look at the 4.

Important note: If the result of a problem is in euros, always round to cents.

Rounding decimal numbers. In this video the student will learn, through visual and didactic examples, to round decimal numbers to the nearest unit, ten, and hundredth. Throughout the video, short exercises are proposed to interact with the student and check understanding. I encourage you to watch it! *Aula chachi.*
`https://www.youtube.com/watch?v=YfLZajSwm4c`

Another important note: If we want to give a result with 2 decimal places, in intermediate steps we will work with more decimal places, at least 3 or 4; otherwise the result will not have the precision we intend. Example:
$A = 9.65$; $B = 6.98$; $C = 4.99$. We want to compute $A \cdot B \cdot C^2$. If we do $A \cdot B$ and round to hundredths we get 67.36, and now multiply by $4.99^2 = 24.90$ we get 1 677.26.

The correct result is 1 677.20, where we only rounded at the end.

4.2 Significant Figures

It is the number of "meaningful" digits used to express an approximate number.

A few examples and you'll understand:

- 2.25 has 3 significant figures;
- 28.049 has 5 significant figures;
- 5.00 has 3;
- 4 000.01 has 6;
- 10 000 we do not know how many significant figures it has; it could be 1, 2, 3, 4, or 5; we must be told to which figure it has been approximated. For this last case, scientific notation can be used to state precisely the number of significant figures, thus:
  - $1 \cdot 10^4$ has 1 significant figure,
  - $1.0 \cdot 10^4$ has 2, and so on up to $1.0000 \cdot 10^4$ which has 5.

Considerations:

- Non-zero digits are always significant.
- Zeros on the left are never significant figures: 0.0002 has 1 significant figure.
- Zeros in between other non-zero digits are always significant: 2004 has 4 significant figures.

More than the number of decimal places, the precision of an approximation is measured by the number of significant figures. No more digits than the situation requires should be used.

Significant Figures (Solved Exercises + 5 KEY Rules). Significant figures, rounding, and scientific notation are very important. In this video you

will learn all the rules and criteria of significant figures with solved exercises. *Javier Delgado*
`https://www.youtube.com/watch?v=ey5EgiPlu0A`

4.2.1 Proposed Activities

1. Copy this table into your notebook and round each number to the indicated number of significant figures:

Number	**1 sig. fig.**	**2 sig. fig.**	**3 sig. fig.**	**4 sig. fig.**
$\sqrt{10}$				
$1/7$				
95 549				
100 000				
30 000				
$3 \cdot 10^4$				
1.9995				
2.000				
20.55				

4.3 Absolute Error and Relative Error

I.– Absolute Error

The absolute error (AE) is defined as AE = |real value − approximate value|.

The vertical bars mean "absolute value" and indicate that the result will always be positive.

Example: We approximate 1/3 of a litre by 0.33 litres.

$$\text{AE} = \left|\frac{1}{3} - 0.33\right| = 0.00333\ldots \approx 0.0033 \text{ litres.}$$

Another example: We approximate 16/6 kg with 2 significant figures (2.7 kg).

$$\text{AE} = \left|\frac{16}{6} - 2.7\right| = |-0.0333\ldots| \approx 0.033 \text{ kg.}$$

One should not use too many significant figures in the absolute error; 2 or 3 is enough. The absolute error has the same units as the quantity being approximated.

Are these errors large or small? The answer is: compared to what? For this purpose, the relative error is defined, which gives us a measure of how large or small the absolute error is.

II.– Relative Error

To compare errors of different magnitudes or numbers, the Relative Error (RE) is defined as:

$$\text{RE} = \frac{\text{AE}}{\text{Real value}},$$

which is often multiplied by 100 to talk about the percentage of relative error.

If the real value is not known, it is replaced by the approximate value (the difference is usually small).

Let us calculate the relative error for the examples above:

$$1^{\text{st}}\ \text{RE} = \frac{0.0033}{1/3} = 0.0099 \Rightarrow 0.99\%\ \text{RE}; \qquad 2^{\text{nd}}\ \text{RE} = \frac{0.033}{8/3} \approx 0.0124 \Rightarrow 1.2\%\ \text{RE}.$$

Now we can say that the 1st approximation has a smaller error than the 2nd, because the relative error is smaller. Relative error (RE) has no units, so errors of different magnitudes or with different units can be compared.

What if the exact value is unknown?

In this case the absolute error cannot be calculated; however, all measuring instruments have a maximum absolute error.

- Bathroom scales that measure in steps of 100 g have a maximum absolute error of 50 g.
- Stopwatches that measure hundredths of a second have a maximum absolute error of 0.005 s, half a hundredth.
- Normal rulers that measure mm have a maximum absolute error of 0.5 mm = 0.05 cm = 0.0005 m.

This is called the **bound of the absolute error**.

Solved Activities

You weigh yourself on a bathroom scale and it reads 65.3 kg; the maximum absolute error is 0.05 kg (50 g). Now we weigh a car on a special scale and it weighs 1 250 kg with a maximum absolute error of 10 kg. Which measurement is more precise?

$$\text{You} \Rightarrow \text{RE} \leq \frac{0.05}{65.3} = 0.00077 \Rightarrow \text{RE} \leq 0.077\%;$$

$$\text{Car} \Rightarrow \text{RE} \leq \frac{10}{1250} = 0.008 \Rightarrow \text{RE} \leq 0.8\%.$$

The bathroom scale is much more precise in this case. However, if we weigh a baby on the same scale and it reads 3.1 kg, the relative error turns out to be $\leq 1.6\%$ (try it), and now the bathroom scale measurement is much less precise.

Thus, the error depends on the precision of the instrument and on the measurement we make with it.

Proposed Activities

1. Prove that 123.45 with AE = 0.005 and 0.12345 with AE = 0.000005 have the same RE.

2. Answer True or False and justify your answer:

   (a) For the same measuring instrument, the error made is smaller the smaller the measurement.

   (b) One cannot compare relative errors of different magnitudes.

   (c) Setting prices like 1.99 €/kg is an attempt to deceive.

   (d) Buying at 1.99 €/kg instead of 2 €/kg means a saving.

   (e) Writing many digits in a result means one is a great mathematician.

   (f) Precision is measured by the number of decimal places.

5 Fractions and Decimals

We will see how to convert a fraction to a decimal and a decimal to a fraction.

5.1 Decimal Expression of a Fraction

Every fraction has a decimal expression obtained by dividing the numerator by the denominator:

$$\frac{a}{b} = a : b.$$

Examples:

$$\frac{3}{25} = 0.12; \quad \frac{68}{99} = 0.686868\ldots; \quad \frac{91}{80} = 1.1375; \quad \frac{177}{90} = 1.9666\ldots$$

As you can see, sometimes the decimal expression is terminating (the remainder becomes 0), and other times it is repeating, with infinitely many decimals among which a block of digits, called the **period**, repeats.

Does this always happen—either terminating or repeating? You will answer yourself when you read what follows.

Let us do $1/17 = 1 : 17 = 0.05882352941\ldots$, which are the digits shown by the calculator; it does not seem to have a period, but might it be possible that it does have one but we cannot see it because it is very long?

Let us start the division. The remainders obtained are $10; 15; 14; 4; 6; \ldots$ As you know, the remainders are less than the divisor, and in this case they can be $1; 2; 3; 4; \ldots; 15$ or 16; 0 cannot appear (we explain later).

Now we ask two questions: What happens if the same remainder appears twice? Must a remainder necessarily repeat at some point?

The answer to the first question: if a remainder repeats, the quotient digit will repeat, and from then on everything will repeat as a period.

The answer to the second question: Yes, it must, for sure! If there are 16 possible remainders and we suppose that all 16 possible have already appeared, what happens when we take out the next one?

You will understand it better with sweets: I have many sweets to distribute among 16 people. I have already given one sweet to each; that is, everyone already has one sweet. I am about to distribute the next one. Will it go to someone who already has one?

This is called the "Pigeonhole Principle" in mathematics, and it is a very powerful tool. Look up something about it.

Put 5 balls into 4 boxes; will there be a box with more than 1 ball?

We hope you have understood. In the worst case, the 17th remainder must coincide with one of the previous ones, the quotient digits will repeat, and therefore the decimal expression is periodic.

You can check that indeed the remainders are 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, 1, 10, ... The worst possible case: the one that makes the number 17 repeats. Usually, it repeats earlier. By the way, the division gives:

$$1 : 17 = 0.05882352941176470588235294117647\ldots$$

a period of only 16 digits!

Although we have seen a particular case, this is a general rule: **The decimal expression of a fraction is either terminating or repeating.** The number of digits in the period of $1/n$ is less than or equal to $n - 1$.

When is it terminating and when repeating?

It is easy. We are given a fraction, for example $\frac{27}{150}$. First we simplify it until we obtain the irreducible fraction: $\frac{27}{150} = \frac{9}{50}$. We look only at the denominator and factor it into primes: $50 = 5 \cdot 10 = 5 \cdot 2 \cdot 5 = 2 \cdot 5^2$. Since the only prime factors are 2 and 5, the decimal expression is terminating.

Let us see the reason: $2 \cdot 5^2$ is a divisor of $2^2 \cdot 5^2 = 100$, a power of 10. We have

$$\frac{1}{50} = \frac{1}{2 \cdot 5^2} = \frac{2}{100} = 0.02,$$

only need to multiply by 9 $\Rightarrow \frac{9}{50} = 0.02 \cdot 9 = 0.18$.

Notice that the number of decimal places is 2, the larger of the exponents of 2 and 5.

For example, $\frac{1}{5^4} = 0.0005$ has 4 decimal places because the larger exponent is 4.

In general, $\frac{1}{2^n \cdot 5^m}$ has a terminating decimal expression and the number of decimal places is the maximum of n and m.

The other case: $\frac{20}{42} = \frac{10}{21}$. We factor 21 into primes: $21 = 3 \cdot 7$. Since there are factors other than 2 and 5, the expression will be repeating.

Let us see: if the expression were terminating, we could write $\frac{10}{21} = \frac{a}{10^n} \Rightarrow 10^n \cdot 10 = a \cdot 21$ with a an integer. But this cannot be! 10 only has factors 2 and 5, and the factors 3 and 7 cannot be cancelled. Since it cannot be terminating, it will be repeating.

If the denominator of an irreducible fraction contains prime factors other than 2 and 5, the decimal expression will be repeating.

5.1.1 Proposed Activities

1. Without performing the division, indicate whether the following fractions have a terminating or repeating decimal expression:

   (a) $\dfrac{21}{750}$

   (b) $\dfrac{75}{21}$

   (c) $\dfrac{11}{99}$

   (d) $\dfrac{35}{56}$

5.2 Half-lines

A **half-line** is the set of numbers either greater than or less than a given number. They are usually denoted using infinity symbols:

- $(a, +\infty) = \{x \in \mathbb{R} \mid x > a\}$ – all numbers greater than a (not including a).
- $[a, +\infty) = \{x \in \mathbb{R} \mid x \geq a\}$ – all numbers greater than or equal to a.
- $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$ – all numbers less than a (not including a).
- $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$ – all numbers less than or equal to a.

Graphically, an arrow is drawn from a towards the indicated infinity; an open or filled dot at a depends on whether the endpoint is included.

Example: The half-line $(4, +\infty)$ is the set of numbers greater than 4.

Example: $(-\infty, -1]$ is the set of numbers less than or equal to -1.

Half-lines are unbounded intervals, so the infinity symbol is always open (it is never reached).

5.3 Neighborhoods

A **neighborhood** is a special way of representing open intervals. The **neighborhood with center a and radius** r, denoted $E(a, r)$ (another common form is $E_r(a)$), is defined as the set of numbers whose distance from a is less than r:

$$E(a, r) = (a - r, a + r).$$

Note that a neighborhood is always an open and bounded interval. An example makes it clearer:

Example: The neighborhood with center 5 and radius 2 consists of the numbers that are at a distance less than 2 from 5. If we think about it, these are the numbers between $5 - 2$ and $5 + 2$, i.e., the interval $(3, 7)$. It is like taking a compass, placing the point at 5, and drawing a circle of radius 2.

Notice that 5 is at the center and the distance from 5 to 7 and from 5 to 3 is 2.

Example: $E(2, 4) = (2 - 4, 2 + 4) = (-2, 6)$.

It is very easy to go from a neighborhood to an interval. Let's do it the other way around.

Example: If we have the open interval $(3, 10)$, how do we write it as a neighborhood?

Find the midpoint: $\frac{3+10}{2} = \frac{13}{2} = 6.5$, which will be the center. Now the radius: $(10 - 3) : 2 = 3.5$ (half the width).

Thus $(3, 10) = E(6.5, 3.5)$.

In general:

$$\text{The interval } (b, c) \text{ is the neighborhood } E\left(\frac{b+c}{2}, \frac{c-b}{2}\right).$$

Example: The interval $(-8, 1) = E\left(\frac{-8+1}{2}, \frac{1-(-8)}{2}\right) = E(-3.5,\ 4.5)$.

6 Problem Solving Using Fractions

We will look at some examples:

i) How many litres are there in 80 bottles, each containing three quarters of a litre?

The first thing to do is to try an example with easier numbers. I have 10 bottles, each 2 litres. Clearly we have 20 litres. What operation did we do? Multiply? Well, we do the same with the original numbers:

$$80 \text{ bottles} \times \frac{3}{4}\frac{\text{litres}}{\text{bottle}} = \frac{80 \cdot 3}{4} \text{ litres} = 60 \text{ litres}.$$

MAKE IT EASIER TO START

(Notice that “bottles” cancel with “bottles” and the final units are litres.)

ii) How many bottles, each holding three eighths of a litre, do I need to package 900 litres?

Again, change the numbers for simpler ones: I want to package 10 litres in 2-litre bottles. Clearly I need 5 bottles (10 : 2). We do the same with our numbers:

$$900 \text{ litres} : \frac{3}{8}\frac{\text{litres}}{\text{bottle}} = 900 : \frac{3}{8} = 900 \cdot \frac{8}{3} = 300 \cdot 8 = 2\,400 \text{ bottles}.$$

Notice that litres cancel and the “bottles” that were dividing in the denominator end up multiplying in the numerator, so the result’s unit is “bottles”.

$$\frac{\text{litres}}{1} : \frac{\text{litres}}{\text{bottle}} = \frac{\text{litres} \cdot \text{bottle}}{\text{litres}} = \text{bottle}.$$

iii) Lluvia earns a certain amount of money per month. She spends 40% of it on the rent, 75% of what remains on bills, and has \$90 left for food. How much does she earn, and how much does she spend on rent and on bills?

First: $40\% = \frac{40}{100} = \frac{2}{5}$ and $75\% = \frac{75}{100} = \frac{3}{4}$.

We solve it in two ways; choose the one you prefer:

a) Graphical method:

Draw a rectangle of 5×4 squares (the denominators). From the 5 equal vertical strips, remove 2 — that is what is spent on rent.

(Here would be the drawing: 5 columns, 4 rows. The first two columns are shaded.)

What remains is divided into 4 equal horizontal parts, and we remove 3 — that is what is spent on bills. We are left with 3 little squares, which represent the \$90 for food. Therefore each little square is 90 : 3 = 30 euros.

She earns: $30 \cdot 20 = 600$ €.

Rent: $30 \cdot 8 = 240$ €. Bills: $30 \cdot 9 = 270$ €.

b) Using fractions:

If from an amount we remove $\frac{2}{5}$, what remains is $\frac{3}{5}$ of it $(1-\frac{2}{5}=\frac{5}{5}-\frac{2}{5})$.

On bills we spend $\frac{3}{4}$ of the remaining $\frac{3}{5}$: $\frac{3}{4}\cdot\frac{3}{5}=\frac{9}{20}$ of the initial amount.

We had $\frac{3}{5}$ and we spent $\frac{9}{20}$; what remains is:

$$\frac{3}{5}-\frac{9}{20}=\frac{12}{20}-\frac{9}{20}=\frac{3}{20}$$

of the initial amount. These $\frac{3}{20}$ are \$90. Therefore $\frac{1}{20}$ is $90:3=30$ €.

The total amount is $\frac{20}{20}=30\cdot 20=600$ €.

On rent: $\frac{2}{5}$ of $600=1200:5=240$ €. On bills: $\frac{3}{4}$ of $(600-240)=\frac{3}{4}\cdot 360=270$ €.

In any case, problems should be checked:

$$\begin{aligned}&40\% \text{ of } 600=0.4\cdot 600=240 \text{ € on rent.}\\&600-240=360 \text{ € left.}\\&75\% \text{ of } 360=0.75\cdot 360=270 \text{ € on bills.}\\&360-270=90 \text{ € left for food.} \quad \text{It works!}\end{aligned}$$

I have	I remove	I am left with
1	2/5	3/5
3/5	3/4 of 3/5 = 9/20	3/5 − 9/20 = 3/20

iv) A ball loses one fifth of its height with each bounce.

1. How many bounces must it make for the height reached to be less than one tenth of the initial height?

2. If after the fourth bounce its height is 12.8 cm, what was the initial height?

First, realise that if it loses one fifth of its height, it keeps four fifths of it. Therefore, with each bounce the height is multiplied by $\frac{4}{5}$.

a) We need to find the smallest integer n such that:

$$\left(\frac{4}{5}\right)^n<\frac{1}{10}=0.1.$$

We test with a calculator:

$$\left(\frac{4}{5}\right)^{10}\approx 0.107>0.1, \qquad \left(\frac{4}{5}\right)^{11}\approx 0.0859<0.1.$$

So 11 bounces are needed.

b) After the fourth bounce the height is the initial height multiplied by $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$.

$$\frac{256}{625}h = 12.8 \implies h = 12.8 \cdot \frac{625}{256} = 31.25 \text{ cm.}$$

v) One fifth of Mariana's gross salary is deducted for income tax (IRPF) and one sixth for Social Security. If she receives \$600 net, what is her gross salary?

We add the two fractions since they refer to the same quantity:

$$\frac{1}{5} + \frac{1}{6} = \frac{6+5}{30} = \frac{11}{30},$$

which is the part deducted from the gross salary. She is left with

$$1 - \frac{11}{30} = \frac{19}{30}$$

of the initial amount. These $\frac{19}{30}$ amount to \$600.

To find the gross salary:

$$600 \cdot \frac{30}{19} \approx 947.37 \text{ €}.$$

Check:

$$1/5 \text{ of } 947.37 = 189.47 \text{ € for IRPF.}$$
$$1/6 \text{ of } 947.37 = 157.90 \text{ € for Social Security.}$$
$$947.37 - 189.47 - 157.90 = 600 \text{ € net.} \quad \text{Good!}$$

There might be a small discrepancy of a few cents due to rounding.

CURIOSITIES. MAGAZINE

Sum of infinitely many fractions

Common sense tells you that if you add infinitely many positive numbers, the sum must be infinite. Well, not necessarily!

We propose a challenge: let's sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

The dots indicate that this never ends; in theory we would have to add and add and keep on adding indefinitely. In practice it cannot be done, but that's what mathematics is for.

Take your calculator and start: 1:2 + 1:4 + 1:8 + 1:16 + 1:32 + 1:64. It gives 0.984375 or, if you are lucky, 63/64 — only 1/64 is missing to reach 1!

Now add 1/128 to the previous result: we get 0.9921875, i.e., 127/128, only 1/128 is missing to reach 1. You must continue; the next numbers to add are 1/256, 1/512, 1/1024, ...

If you have noticed, we get closer and closer to 1. Okay, we will never arrive, but if we wanted to assign a value to the infinite sum above, what value would you give it?

Mathematicians give it the value 1.

Observe. You have a square sheet of paper of area 1. You cut it in half, put the cut piece on the table and keep the uncut piece in your hand. You cut the piece in your hand in half again, and place the cut piece on the table. And so on. . . You add up the pieces of paper on the table. Could the sum ever exceed 1? No, obviously; they are pieces of a paper of area 1. Will you ever have the whole paper on the table? Each time you have less paper in your hand and more on the table, but by cutting in half, you will never have it all. However, mathematicians say that at infinity that sum equals 1.

Now we have a pizza and we are going to eat it in "thirds of thirds", i.e., first 1/3, then 1/3 of 1/3, then 1/3 of 1/3 of 1/3, and so on...

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

What do you think this sum equals? (It equals 1/2.)

SUMMARY

Priority of operations	1st Inner parentheses, 2nd Powers and roots, 3rd Multiplication and division, 4th Addition and subtraction. Example: $10 - 5 \cdot (4 - 3 \cdot 2^2) = 50$
Sign of addition	$(+) + (+) = (+)$ you add; $(-) + (-) = (-)$ you add. $(+) + (-) =?$ takes the sign of the larger absolute value. Example: $-7/3 - 8/3 = -15/3 = -5$; $-12/5 + 8/5 = -4/5$
Sign of multiplication and division	If they have the same sign, result is positive: $(+) \cdot (+) = (-) \cdot (-) = (+)$. If they have opposite signs, result is negative: $(+) \cdot (-) = (-) \cdot (+) = (-)$. Example: $-4 \cdot (-10) = +40$; $+2 \cdot (-15) = -30$
Rational number	A number r is rational if it can be written as $r = a/b$ with a, b integers and $b \neq 0$. Examples: $2; 3/8; -7/2$ are rational. Also 0.125 and $2.6777\ldots$ $\sqrt{2}$ and π are not.
Irreducible fraction	Obtained by dividing numerator and denominator by the same number. Numerator and denominator are coprime. Example: $360/840 = 3/7$, the latter is irreducible.
Equivalent fractions	Fractions are equivalent if they have the same decimal value. Two equivalent fractions represent the same rational number. Their cross products are equal. Examples: $3/4 = 6/8$; $4/8 = 15/30$ (equivalent: $3 \cdot 20 = 4 \cdot 15$)
Ordering fractions	Reduce to a common denominator, find their decimal values, or use logic and the trick $a/b < c/d$ if $ad < bc$ for positive numbers. Examples: $3/4 < 9/10$; $4/5 > 10/20$, among other reasons.
Representation on the number line	If necessary, convert to mixed form. For $n + a/b$, divide the unit from n to $n + 1$ into b equal parts and take a. For $-n - a/b$, divide the unit from $-n$ to $-n - 1$ into b equal parts and count a starting at $-n$.

SUMMARY (continued)

Errors	Absolute error: $EA = \lvert\text{real value} - \text{approximate value}\rvert$. Relative error: $ER = \dfrac{EA}{\lvert\text{real value}\rvert}$; multiply by 100 for % ER. Example: $2/3 \approx 0.7 \Rightarrow EA \approx 0.033$ $\Rightarrow ER \approx \dfrac{0.033}{2/3} \approx 0.050 \Rightarrow 5\%$
Fractions and decimals	The decimal expression of a fraction is always either terminating or repeating. Terminating if the denominator only has 2 and/or 5 as prime factors; repeating otherwise. Example: $3/40 = 0.075$ terminating; $5/12 = 0.41666...$ repeating.
From decimal to fraction	Terminating decimal: write the number without the decimal point over 1 followed by as many zeros as decimal digits. Repeating decimal: multiply N by powers of 10 to obtain two numbers with the same decimal part, subtract and solve for N. Example: $3.175 = 3175/1000 = 127/40$. $N = 2.0333...$; $100N - 10N = 183$; $90N = 183 \Rightarrow N = 183/90 = 61/30$.
Real numbers	Every finite or infinite decimal expression is a real number, and vice versa. Examples: $0.333333; \pi; \sqrt{2}$
Open interval	Interval in which the endpoints do not belong. $(2, 7) = \{x \in \mathbb{R} \mid 2 < x < 7\}$.
Closed interval	Endpoints DO belong. $[-2, 2] = \{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$.
Semi-open (or semi-closed) intervals	Interval with one open endpoint and one closed. Example: $[-8, 0) = \{x \in \mathbb{R} \mid -8 \leq x < 0\}$.
Neighborhoods	Special way to express an open interval: $E(a, r) = (a - r, a + r)$.

EXERCISES

1. Calculate step by step:

$$(-5+4\cdot(-2)+7):(7-(3-4)\cdot(-1))$$

2. Order from smallest to largest:

$$\frac{8}{9},\quad -\frac{8}{9},\quad \frac{4}{5},\quad \frac{38}{45},\quad \frac{77}{90},\quad -\frac{9}{8}$$

3. Reason which fraction is larger:

$$\text{a) } \frac{102}{101} \text{ and } \frac{98}{99} \qquad \text{b) } \frac{98}{99} \text{ and } \frac{97}{98} \qquad \text{c) } -\frac{102}{101} \text{ and } -\frac{103}{102}$$

4. Show that $4.999\ldots = 5$. Generalise: What is the value of $n.999\ldots$?

5. Use a calculator or a spreadsheet and reflect on the hierarchy of operations:

   (a) $2.34567/361.8792 + 7.835261 * 4319.8872 - 63.52$

   (b) $2.34567/(361.8792 + 7.835261) * (4319.8872 - 63.52)$

   (c) $(2.34567/361.8792) + (7.835261 * 4319.8872) - 63.52$

   (d) $(2.34567/361.8792) + 7.835261 * (4319.8872 - 63.52)$

6. Convert to mixed form: $\frac{16}{9}$; $\frac{152}{6}$; $-\frac{17}{5}$; $-\frac{23}{4}$.

7. Represent exactly on the number line:

$$\frac{760}{240};\qquad 3.125;\qquad -\frac{46}{14};\qquad -2.1666\ldots$$

8. Simplify:

$$\text{a) } \frac{2\cdot 7\cdot 15}{21\cdot 10} \qquad \text{b) } \frac{10+6}{10-2} \qquad \text{c) } \frac{2\cdot 3+4}{2\cdot 5+10}$$

9. Find the fraction that falls exactly halfway between $3/2$ and $9/4$ on the number line.

   Hint: the arithmetic mean $\frac{a+b}{2}$

   Represent the three fractions on the number line.

10. The harmonic mean is defined as $H(a,b) = \frac{1}{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)}$, the reciprocal of the arithmetic mean of the reciprocals.

(a) Show that $H(a,b) = \dfrac{2ab}{a+b}$.

(b) Find $H\left(\dfrac{3}{2}, \dfrac{11}{3}\right)$.

11. Find the inverse fraction of $3 + \dfrac{4}{5} : \dfrac{6}{10}$.

12. Operate and simplify:
$$\frac{4}{6} \cdot \frac{10}{7} \cdot \frac{5}{14} \cdot \frac{12}{2}$$

13. Solve step by step:
$$\frac{\frac{3}{5} - \frac{2}{5} \cdot \frac{4}{6}}{\frac{3}{5} \cdot \left(\frac{1}{6} - 2\right)}$$

14. Find two thirds of one sixth of 80% of 900.

15. Find the number such that its four-thirds equal 520.

16. How many $\frac{3}{8}$-litre jars can be filled with 12 litres?

17. Find the fraction by which 450 must be multiplied to obtain 720.

18. If 100 inches are 254 cm:

    (a) Find the length in cm of a television if the height is 19.2 inches and the ratio length/height = 4/3.

    (b) The same, but now the ratio is 16/9.

19. In a class, 77.777... % of the pupils pass, and there are more than 30 but fewer than 40 pupils. How many pupils are there, and how many pass?

20. Three pilgrims decide to undertake an 8-day journey. The first pilgrim contributes 5 loaves of bread, the second contributes 3 loaves, and the third contributes none but promises to pay his companions at the end of the journey for the bread he has eaten. Each day at lunchtime they take a loaf from the bag, divide it into three pieces, and each pilgrim eats one piece. When they reach their destination, the pilgrim who brought no bread takes out 8 coins and gives them to his companions: 5 coins to the one who had put in 5 loaves, and 3 coins to the one who had contributed 3 loaves. Can you explain why this division of the coins is not fair? What would be the fair division? (Problem from the Albacete Olympiad. Note: one must take into account not the loaves one put in, but what one really contributed (the loaves put in minus the loaves eaten).)

EXPERIMENT, PLAY WITH THE PROBLEM

21. Approximate the numbers 32 567 and 1.395 to 2 significant figures, and state in which case the smaller relative error is made.

EXERCISES (1–10)

1. Calculate step by step:

$$(-5 + 4 \cdot (-2) + 7) : (7 - (3 - 4) \cdot (-1))$$

2. Order from smallest to largest:

$$\frac{8}{9}, \quad -\frac{8}{9}, \quad \frac{4}{5}, \quad \frac{38}{45}, \quad \frac{77}{90}, \quad -\frac{9}{8}$$

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CURIOSITIES. MAGAZINE

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We propose a challenge: let's sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$$

where each fraction is half of the previous one. The dots indicate that this never ends; in theory we would have to add and add and keep on adding indefinitely. In practice it cannot be done, but that's what mathematics is for.

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Neighborhoods	Special way to express an open interval: $E(a, r) = (a - r, a + r)$.

Problem solving with fractions (additional examples)

iv) A ball loses one fifth of its height with each bounce.

1. How many bounces must it make for the height reached to be less than one tenth of the initial height?
2. If after the fourth bounce its height is 12.8 cm, what was the initial height?

First, realise that if it loses one fifth of its height, it keeps four fifths of it. Therefore, with each bounce the height is multiplied by $\frac{4}{5}$.

a) We need the smallest integer n such that $\left(\frac{4}{5}\right)^n < \frac{1}{10} = 0.1$. Testing: $\left(\frac{4}{5}\right)^{10} \approx 0.107 > 0.1$, but $\left(\frac{4}{5}\right)^{11} \approx 0.0859 < 0.1$. So 11 bounces are needed.

b) After the fourth bounce: $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$ of the initial height.

$$\frac{256}{625}h = 12.8 \implies h = 12.8 \cdot \frac{625}{256} = 31.25 \text{ cm}.$$

v) One fifth of Mariana's gross salary is deducted for income tax (IRPF) and one sixth for Social Security. If she receives \$600 net, what is her gross salary?

Add the fractions (same base):

$$\frac{1}{5} + \frac{1}{6} = \frac{6+5}{30} = \frac{11}{30},$$

part deducted. She keeps $1 - \frac{11}{30} = \frac{19}{30}$ of gross, which equals \$600.

$$\text{Gross} = 600 \cdot \frac{30}{19} \approx 947.37€.$$

Check: $\frac{1}{5}$ of $947.37 = 189.47$ (IRPF); $\frac{1}{6}$ of $947.37 = 157.90$ (S.S.); remainder 600. (A few cents rounding difference may occur.)

You could also solve it using a graphical method: a 5×4 rectangle, shading spent parts, leaving 3 squares = 90 €, etc. (see earlier).